

Unraveling hidden hierarchies and dual structures in an integrable field model

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An integrable field theory, due to path-independence on the space-time plain, should yield together with infinite sets of conserved charges also similar dual charges, determining the boundary and defect contributions. On the example of the nonlinear Schrödinger equation we unravel hidden hierarchies and dual structures and show the complete integrability through a novel Yang-Baxter equation at the classical and quantum level with exact solution.

The Nöther's theorem (NT) connects the symmetries of a model to its invariant quantities. In a mechanical system with finite degrees of freedom such quantities are the first integrals conserved in time. For the integrability and exact solvability of a system the number of conserved charges should match with its degrees of freedom [1]. This definition is usually generalized straightforwardly to the field theory (FT) defining the integrability of a field model through an infinite set of charges conserved in time. The NT is also tuned to fit this definition, stating that the model should have infinite number of symmetries with one to one correspondence to the conserved charges [1]. However, in reality neither the integrable FT nor the NT insists on such inherently noncovariant properties, which are set perhaps by tradition and not by necessity. In fact, the NT relates each symmetry of a field model to an integrated quantity invariant along any chosen direction in the space-time manifold [2] and similarly an integrable FT, based on the flatness condition involving both of its Lax operators, allows a path-independent formulation on the x, t plain with no emphasis on the direction of evolution [3]. However, as mentioned above, the conventional approach in integrable FT focuses mostly on space configurations at a fixed time generating an infinite set of charges conserved in time, through the space-Lax operator, which acts as an infinitesimal shift operator (ISO) along the x -direction. This traditional formulation however, as we show, works well only for models with periodic or vanishing boundary condition (BC) and with no internal defect, but fails for more general BC and in the presence of defect points, where the usual charges no longer remain conserved [4, 5]. In dealing with the time-dependent boundary conditions, relevant in real experiments [6], the traditional scattering data along the space direction is found also to be insufficient, prompting the inclusion of scattering processes in both space and time directions [4]. These explicit conflicts with the established FT force us to rethink about the limitations of the accepted theory and propose a dual approach to complement the existing results by unraveling a wealth of hidden structures in the integrable FT. In particular, working on the example of the nonlinear Schrödinger (NLS) equation we obtain an infinite set of dual charges conserved in space, generated by the time-Lax operator acting as an ISO along the t -direction. This addresses a rather philosophical question, asking how many infinite sets of conserved charges one needs for the integrability of a FT, having infinite degrees of freedom. By finding an additional set of dual charges $J_k^{(m)}$, $m = 2$, $k = 1, 2, \dots$, we could enlarge the known set of conserved charges $c_k^{(n)}$, $n = 1$, $k = 1, 2, \dots$ for the NLS model by another countable infinity. More encouragingly, we find that, the contributions from general BC and the defect point can be given explicitly by these dual charges, which are missed in traditional treatments. Extending to higher Lax pairs one could obtain further hidden charges with $n = 2, 3, \dots$ along with their duals with $m = 1, 3, 4, \dots$, generating newer integrable hierarchies covering all possible cases (see Fig. 1), which are ignored surprisingly in the literature.

The hardest task however is to show the complete integrability and exact solvability of the system by establishing the mutual commutativity of dual charges, in analogy with that for the usual charges, shown by the celebrated Yang-Baxter equation (YBE) linked to the space-Lax operator and solved by the well known Bethe ansatz (BA) [8]. For this we conjecture a *fixed space* Poisson bracket (PB) and the related CR for the NLS system, which could derive a novel hierarchy of integrable equations from the dual charges and at the same time construct a dual YBE linked to the time-Lax operator and solve the model exactly by the BA.

The integrable NLS equation $iq_t = q_{xx} + 2(q^*q)q$, for the complex field $q(x, t)$ together with its complex conjugate can be obtained from the compatibility of linear Lax equations

$$\Phi_x(\lambda) = U_1(\lambda)\Phi(\lambda), \quad \Phi_t(\lambda) = V_2(\lambda)\Phi(\lambda), \quad \Phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda)), \quad (1)$$

involving space and time Lax operators $U_n(\lambda)$ and $V_m(\lambda)$, with $n = 1$ for the simplest case with space-variable $x = x_1$ and $m = 2$ for the next higher time variable $t = t_2$. In the conventional approach the generating function $\ln \phi_1(x, \lambda) = i \int_{-\infty}^x dx' \rho(x', \lambda)$ for the densities $\rho(x, \lambda) = \sum_k \rho_k(x)(2\lambda)^{-k}$ of charges is considered for vanishing

BC using the simplest space-Lax operator $U_1(\lambda)$. Therefore from the first Lax equation in (1) one can derive the Riccati equation for $\frac{\phi_2}{\phi_1} \equiv \Gamma(x, \lambda) = \sum_k \Gamma_k(x)(2\lambda)^{-k}$, solving which one obtains infinite number of conserved charges $c_k = \int_{-\infty}^{+\infty} dx q \Gamma_k(x)$, $k = 1, 2, \dots$, well known for the NLS model [3]. The mutual commutativity of the charges $[c_k, c_l] = 0$, $k \neq l$ is proved by the classical and quantum YBE involving Lax operator $U_1(\lambda)$ and the rational R -matrix, while NLS equation can be obtained from Hamiltonian $H = c_3$ using the usual PB with canonical momentum q^* [3]. The known NLS hierarchy is obtained from higher charges $H = c_{m+1}$, compatible with the flatness condition of the Lax pairs (U_1, V_m) , $m = 3, 4, \dots$ for time $t = t_m$ (see Fig. 1).

For exploring the dual objects we follow a path parallel to the conventional one, replacing however x by t and $U_1(\lambda)$ by the time-Lax operator $V_2(\lambda)$ for the NLS equation. This results from the second Lax equation in (1), describing the scattering process along the time-direction at fixed x , a generating function $\ln \phi_1(t, \lambda) = i \int_{-\infty}^{\infty} dt' j(t', \lambda)$ for densities $j(t, \lambda) = \sum_k j_k(t)(2\lambda)^{-k}$, of the dual charges J_k , as $j(t, \lambda) = V_{11} + V_{12}\Gamma(t, \lambda)$, involving the time-Lax operator and $\Gamma(t, \lambda) = \sum_k \tilde{\Gamma}_k(2\lambda)^{-k}$. Solving the associated time-Riccati equation recursively for Γ_k we can construct an infinite set of dual charges $J_k = \int_{-\infty}^{\infty} dt j_k(t)$, $k = 1, 2, \dots$, for the NLS equation with vanishing BC on the time-axis (see Appendix). Similar idea was used earlier for obtaining nonlocal charges in the sine-Gordon field model [7].

Importantly, for considering models beyond periodic or vanishing BC in the space interval $[x_0, x_1]$, or the models with a defect at point x_d , the traditional treatment engaging only the space-Lax operator becomes insufficient and the inclusion of the time-Lax operator essential. One finds that, the actual conserved charges are not the traditional ones mentioned above, but more general ones as

$$C_k = c_k + I_k^{BC}, \quad I_k^{BC} = J_k(x_0, t) - J_k(x_1, t), \quad k = 1, 2, \dots, \quad (2)$$

with c_k as the traditional charges, while I_k^{BC} given through dual charges J_k is the boundary contribution, or similarly

$$C_k = c_k^- + c_k^+ + I_k^{def}, \quad I_k^{def} = J_k(x_d^+) - J_k(x_d^-), \quad k = 1, 2, \dots \quad (3)$$

with vanishing BC but with a defect contribution I_k^{def} given again through J_k at $x_d^\pm = (x_d \pm \epsilon)_{|\epsilon \rightarrow 0}$, where c_k^\pm are usual charges for the fields confined to the left (right) from the defect point. It is remarkable that the dual charges J_k , which are derived here but traditionally ignored, determine actually the boundary and defect contributions.

Switching over to the dual picture we can show similarly that, in more general case of time-interval $[0, T]$ at fixed space x , the dual charges conserved in space generalise to

$$\tilde{J}_k = J_k + T_k^{BC}, \quad T_k^{BC} = c_k(0) - c_k(T), \quad k = 1, 2, \dots, \quad (4)$$

with time-boundary contribution T_k^{BC} and in models with *time*-defect at moment $t = t_d$ to

$$\tilde{J}_k = J_k^+ + J_k^- + T_k^{def}, \quad T_k^{def} = c_k(t_d^+) - c_k(t_d^-), \quad k = 1, 2, \dots, \quad (5)$$

with time-defect contribution T_k^{def} , where J_k^\mp are dual charges in the semi-infinite time intervals before and after the defect-moment t_d .

However, it is natural to ask: how far we can stretch this concept of duality. Can dual charges J_k , $k = 1, 2, \dots$, generate a new hierarchy of integrable equations including the NLS equation, mimicking the standard approach but using some unusual canonical bracket? Defining the dual canonical momentum as $\tilde{p} = \frac{\delta L}{\delta q_x}$ we derive from the NLS Lagrangian $\tilde{p} = q_x^*$, $\tilde{p}^* = q_x$ and conjecture an equal-space canonical bracket $\{q(x, t), \tilde{p}(x, t')\} = \delta(t - t')$, $\{q^*(x, t), \tilde{p}^*(x, t')\} = \delta(t - t')$, $\{q(t), q^*(t')\} = \{\tilde{p}(t), \tilde{p}^*(t')\} = 0$. Using this dual PB and taking $J_2 = H$ as the Hamiltonian we derive $\tilde{p}_x^* = \{\tilde{p}^*, J_2\} = iq_t - 2|q|^2 q$, which by putting $\tilde{p}^* \equiv q_x$ would yield the same NLS equation associated with the Lax pair (U_1, V_2) . From next higher order however our dual PB starts producing new equations, e.g. for $H = J_3$ one gets a linear equation $q_t = q_{x_2}$ related to the pair (U_2, V_2) and for $H = J_4$, an intriguing integrable equation

$$iq_{xt} = q_{x_3} + 2(q_x^* q - q^* q_x), \quad (6)$$

associated with the Lax pair (U_3, V_2) and so on, which generates a novel integrable hierarchy represented by the flatness condition of the infinite series of Lax pairs (U_k, V_2) , $k = 1, 2, \dots$ as a dual to the NLS hierarchy (see Fig. 1).

Encouraged by the success with our conjecture we set a harder task to establish the complete integrability of the system by showing mutual independence of dual charges $[J_k, J_l] = 0$, $k \neq l$ through a novel dual Yang-Baxter relation both in classical and quantum cases, where the space-Lax operator $U_1(\lambda)$, appearing in the standard YBE for the

NLS model [8] is replaced by its time-Lax operator $V_2(\lambda)$, while the classical $r(\lambda - \mu)$ and the quantum $R(\lambda - \mu)$ matrices remain the same well known rational matrices (see Appendix).

However, proving the validity of these dual YBEs becomes exceedingly difficult, due to much more involved structure of $V_2(\lambda)$ for the NLS model compared to that of $U_1(\lambda)$. Consequently, in place of just *two* nontrivial relations appearing in known YBEs [3, 8], *ten* such equations arise in the dual YBEs, apart from the ultralocal conditions. However miraculously the dual PB we propose solves all classical relations exactly, proving the dual YBE for the classical NLS model. More significantly, for the quantum NLS model the corresponding equal-space commutation relations (CR) can be given for the field operators discretize along the time axis as $[q^j(x), q^{*k}(x)] = \hbar \Delta_{(t)}^{-1} \delta_{jk}$, $[q^j(x), q^{*k}(x)] = 0$, $\Delta_{(t)}$ being the lattice constant in discrete time. The lattice regularized V_2^j operator together with the rational quantum R -matrix and the proposed CR, solve fortunately all nontrivial commutation relations in the dual quantum YBE, exactly upto order $O(\Delta_{(t)})$, which is however sufficient for the QFT model of NLS we are interested in (see Appendix).

Recall that the YBE solutions are the key for exactly solving the eigenvalue problem (EVP) for quantum charges [8]. Using the same idea we can formulate an algebraic Bethe ansatz (ABA) for the exact EVP solution of the dual quantum charges, by constructing a dual monodromy matrix $S(\lambda) = \prod_j V^j(\lambda)$ with the *creation* and *annihilation* operators defined through its operator elements as $B(\lambda) = S_{12}(\lambda)$ and $C(\lambda) = S_{21}(\lambda)$ respectively, while its diagonal elements: $\tau(\lambda) = \text{tr} S(\lambda)$ with $\ln \tau(\lambda) = \sum_k J_k(2\lambda)^{-k}$, are related to dual quantum charges J_k , $k = 1, 2, \dots$. The EVP: $\tau(\lambda)|N\rangle = \Lambda(\lambda)|N\rangle$ can be solved exactly in perfect analogy with the standard ABA formulation [8], since the R -matrix contribution from our dual YBE remains the same. Only the vacuum expectation of $U_1(\lambda)$ appearing in eigenvalue $\Lambda(\lambda)$ in the usual case, should be replaced by the corresponding value for the time-Lax operator $V_2(\lambda)$ in the dual case. We would like to conclude in general that, the NLS field model possesses rich hidden structures and

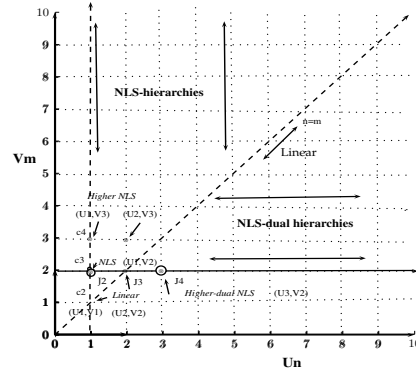


FIG. 1: Complete NLS hierarchy with Lax pair (U_n, V_m) , $n, m = 1, 2, \dots$. Vertical lines are the hierarchies with infinite charges $c_{m+1}^{(n)}$ conserved in time t_m and horizontal lines are the dual-hierarchies with infinite dual charges $J_{n+1}^{(m)}$ conserved in space x_n . The vertical dashed line corresponds to the known NLS hierarchy, while the horizontal solid line to the dual hierarchy found here.

multiple integrable hierarchies beyond the known ones, associated with a double-infinite series of Lax pairs (U_n, V_m) with flatness condition $U_n t_m - V_m x_n + [U_n, V_m] = 0$, covering all possible cases $(n, m) = 1, 2, \dots$ as shown in Fig. 1. The diagonal line with $n = m$ represent linear equations since $[U_n, V_n] = 0$ due to $U_n \sim V_n$, while the vertical lines with $m = 1, 2, \dots$ for each fixed n , represent integrable hierarchies with a set of infinite charges $c_{m+1}^{(n)}$ conserved in time t_m , of which only $n = 1$ corresponds to the known NLS hierarchy. The hierarchy with $n = 2$ can be obtained by replacing the variables $t \rightarrow x_2$, $x_m \rightarrow t_m$ in the explicit form of the dual hierarchy found here. The horizontal lines in Fig. 1 with $n = 1, 2, \dots$ at fixed m , represent on the other hand integrable dual hierarchies with a set of infinite dual charges $J_{n+1}^{(m)}$ conserved in space x_n . $m = 1$ case corresponds to the equations, where the variables in the known NLS hierarchy are replaced as $x \rightarrow t_1$, $t_n \rightarrow x_n$, while the case with $m = 2$ corresponds to the dual hierarchy found above including (6). All common crossing points of vertical and horizontal lines must yield the same equations derivable from both usual charge and their duals, as we have witnessed above in deriving the NLS equation associated with the Lax pair U_1, V_2 at $n = 1, m = 2$, from our dual charge $J_2^{(2)}$ as well as from the NLS Hamiltonian $c_3^{(1)}$.

In summary, using time as dual to the space variable we find explicitly an infinite set of dual charges $J_k^{(2)}$, $k = 1, 2, \dots$ in addition to the set of usual charges $c_k^{(1)}$ in an integrable field model, on the example of the NLS equation. The dual charges generated by the time-Lax operator V_2 not only determine the defect and the boundary contributions missed

in a traditional approach, but also ensure the complete integrability through a dual Yang-Baxter equation, both in classical and quantum cases with the use of a novel canonical momentum and an equal-space canonical commutator. Similarly, the contributions of time-boundaries and time-defect to the general dual charges are determined in turn by the usual set of charges, due to the space-time duality. It is found that, in general Lax pairs (U_n, V_m) with all possible combinations of $(n \in \mathbf{N}, m \in \mathbf{N})$ should generate multiple-infinite sets of integrable hierarchies shown in Fig. 1, only one of which (with $(n = 1, m \in \mathbf{N})$) is known, while a few (with $(m = 2, n \in \mathbf{N})$, $(m \in \mathbf{N}, n = 2)$, $(m = 1, n \in \mathbf{N})$) are found by us including the equation (6). These are the main results obtained here apart from the exact solution of the NLS quantum field model along the dual direction by algebraic Bethe ansatz. The present investigation not only exposes the insufficiency in the established integrable field theory, but also complements it and opens up new directions of research in integrable classical and quantum systems, like models with time defects and time boundaries as dual to such popular problems on the space axis, exploring the NLS multiple-hierarchy found here for their exact soliton solutions, Hamiltonian structure, YBE etc [10], new class of exactly solvable spin chains along the dual axis using different realizations of the quantum time Lax operator, integrable models on an arbitrary path on a 2D plane construction of novel quantum algebras as dual to the known Hopf algebras in integrable systems [9] etc.

Appendix

I. Infinite set of dual charges

Using the matrix elements $V_{11} = 2\lambda + |q|^2$, $V_{12} = V_{21}^* = 2\lambda q - iq_x$ of the time-Lax operator $V_2(\lambda)$ for the NLS equation, we can derive its dual charges $J_k = \int_{-\infty}^{\infty} dt j_k$, with $j_k = (-iq_x \tilde{\Gamma}_k + q \tilde{\Gamma}_{k+1})$, $k = 1, 2, \dots$, through the solution Γ_k of the time-Riccati equation $i\Gamma_{kt} = \Gamma_{k+2} - 2|q|^2 \Gamma_k + \sum_j (q\Gamma_j \Gamma_{k+1-j} - iq_x \Gamma_j \Gamma_{k-j})$ with densities j_k as $j_1 = i(q_x^* q - q^* q_x)$, $j_2 = iq_t^* q + q_x^* q_x + |q|^4$, $j_3 = q_t^* q_x$, $j_4 = iq_{xt}^* q_x + q_t^* q_t - i|q|^2(q^* q_t - q_t^* q) - 2|q|^2(q_x^* q_x + (q^{*2} q_x^2 + q_x^{*2} q^2))$, and so on.

II. Dual classical YBE

The dual YBE of the NLS model is linked to the time-Lax operator $V_2(\lambda)$ with the same rational r-matrix as $\{V(\lambda, x, t_1) \otimes V(\mu, x, t_2)\} = [r(\lambda - \mu), V(\mu, x, t_1) \otimes I + I \otimes V(\mu, x, t_1)]\delta(t_1 - t_2)$. In this matrix relation ten equations remain nontrivial, which however using the dual PB can be reduced into only two coinciding or related groups with values $-iq$ and $-2i(\lambda + \mu)$ multiplied by $\delta(t_1 - t_2)$ on both sides of the equation proving the dual YBE.

III. Dual quantum YBE

The dual YBE for the NLS quantum field model with regularized time Lax operator V^j and the known rational R-matrix is

$$R(\lambda - \mu)V^j(\lambda) \otimes V^j(\mu) = (I \otimes V^j(\mu))(V^j(\lambda) \otimes I)R(\lambda - \mu), \quad j = 1, 2, \dots, N.$$

In the lattice regularization $V^j(\lambda) = I + i\Delta_{(t)}V_2(\lambda)$, $|\Delta_{(t)}| \rightarrow 0$, the field operators $q(x, t)$, $q_x^*(x, t)$ in the known time-Lax operator $V_2(\lambda)$ are replaced by their semi-discretize variants $q^j(x)$, $q_x^{*j}(x)$. This results to ten nontrivial operator equations containing terms of order $O(\Delta_{(t)})$ apart from higher order terms. For example, in element (11, 12): $\eta(V_{11}^j(\lambda)V_{12}^j(\mu) - V_{11}^j(\mu)V_{12}^j(\lambda)) + (\lambda - \mu)(V_{11}^j(\lambda)V_{12}^j(\mu) - V_{12}^j(\mu)V_{11}^j(\lambda))$, the first term is nontrivial: $-2\eta\Delta_{(t)}(\lambda - \mu)q^j$. However the second term containing $i\Delta_{(t)}^2(\lambda - \mu)[q^{j\dagger}q^j, q_x^j]$, cancels it exactly by using the proposed dual CR. Similar picture is repeated for other terms inducing exact cancellation of all terms upto order $O(\Delta_{(t)})$. The ultralocal condition $V^j(\lambda) \otimes V^k(\mu) = V^k(\lambda) \otimes I(I \otimes V^j(\mu))$, $k \neq l$. follows from the canonicity of the dual CR.

The author is indebted to Prof. P. Mitra for stimulating discussions.

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